For example, by assuming that

$$\operatorname{Stab}[A, B] = \frac{100}{\max\{\sqrt{\varkappa(\mathscr{H})}, \sqrt{1 + \|K\|^2}\}}$$

and making use of the fact that $\varkappa(\mathscr{H}) \geq 1$, $1 + \|K\|^2 \geq 1$, we shall always have Stab [A, B] $\leq$ 100 characterizing the "degree of stabilizability" as if it were as a percentage. Other proposals are also possible for the form of the equation expressing Stab [A, B] in terms of $\varkappa(\mathscr{H})$ and $\|K\|$.

The necessity of introducing numerical characteristics for the degree of stabilizability, controllability, detectability, and observability became clear in the process of analyzing the set of equations by means of numerical methods of linear algebra giving a result with a guaranteed estimate of accuracy. A review of problems arising in developing these methods has been given in [6].

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ASYMPTOTICS OF A VELOCITY FIELD AT CONSIDERABLE DISTANCES FROM A SELF-PROPELLED BODY

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Stationary flow is considered for a viscous incompressible liquid outside a finite body in a three-dimensional space. Velocity distribution is prescribed at the surface of the body for a liquid with zero overall flow rate over this surface. At infinity the velocity vector tends toward a zero constant vector. External mass forces may act on the liquid decreasing quite rapidly with distance from the body. It is required that the total pulse applied to the liquid by the boundary of the body and by mass forces equals zero. The conditions listed form a boundary problem for Navier–Stokes equations which we call the problem of pulse-free flow or the problem of flow around a self-propelled body. Asymptotics are constructed for the solution of this problem at considerable distances from the body assuming that this solution exists. These asymptotics have marked differences from those for solving the classical problem of flow around a towed body [1-3].

1. Statement of the Problem. We formulate the problem of pulse-free flow around a body by a viscous liquid. Let Σ be a smooth closed surface in $\mathbf{R}^3$, and Ω be the external surface in relation to the Σ region. We consider in this region a stationary set of Navier–Stokes equations and the continuity

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$$\Delta \mathbf{u} - 2\,\partial \mathbf{u}/\partial x_1 - \nabla p = 2\mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{g}(x),\ \nabla\cdot\mathbf{u} = 0. \tag{1.1}$$

Equation (1.1) is written in dimensionless variables; as scales of length, velocity, and pressure values of $2\nu/V_\infty$, V_∞, and $\rho V_\infty^2/2$ are selected respectively. Here V_∞ = const > 0 is flow velocity modulus at infinity; ρ is liquid density; ν is kinematic viscosity coefficient; $\mathbf{u}$ is deviation of the dimensionless liquid velocity from the flow velocity $\mathbf{e}_1 = (1, 0, 0)$; $\mathbf{g}$ is density of external mass forces. Function $\mathbf{g}$ is assumed to be smooth in region Ω and finite (considered in Sec. 5 is the case of function $\mathbf{g}$ with an uncompact carrier, but with a condition of its rapid decrease with $r = |x| \to \infty$).

In system (1.1) the boundary conditions are linked:

$$\mathbf{u}|_\Sigma = \mathbf{w}(x); \tag{1.2}$$

$$\mathbf{u} \to 0 \text{ for } r \to \infty, \tag{1.3}$$

where $\mathbf{w} + \mathbf{e}_1$ is the prescribed velocity distribution at the boundary of the body. Furthermore, we shall assume that the overall flow rate of liquid over the boundary of the body equals zero:

$$\oint_\Sigma \mathbf{w}\cdot\mathbf{n}\, d\Sigma = 0 \tag{1.4}$$

($\mathbf{n}$ is the unit vector of the normal to the boundary region Ω). If surface Σ does not move and is impenetrable, then $\mathbf{w} = -\mathbf{e}_1$. If also $\mathbf{g} = 0$, then relationships (1.1)-(1.3) form a classical problem of flow for Navier–Stokes equations. A considerable number of works of an analytical nature have been devoted to studying this problem (see [1-3] and the literature cited there), and there are also numerous studies in which it is resolved numerically. It is well known that in this case the resistance force operating on the body from the direction of the liquid differs from zero. Thus, in order to realize a stationary regime for flow around a body, it is necessary to restrain external forces in the stream. Therefore, the problem of flow should be called a problem of flow around a towed body.

We are interested in the problem of flow around a self-propelled body. The condition of self-propulsion means that the total impulse applied to the liquid by the boundary of the body around which there is flow and mass forces equal zero. Mathematically this condition is expressed by the equation

$$-F \equiv \oint_\Sigma [P\mathbf{u}\cdot\mathbf{n} - 2\mathbf{u}(\mathbf{u} + \mathbf{e}_1)\cdot\mathbf{n}]\, d\Sigma + \int_\Omega \mathbf{g}\, dx = 0. \tag{1.5}$$

Here $P\mathbf{u}$ is the stress tensor corresponding to the velocity field $\mathbf{u}$ and pressure p; elements of this tensor have the form $(P\mathbf{u})_{ij} = -p\delta_{ij} + \partial u_i/\partial x_j + \partial u_j/\partial x_i$ (i, j = 1, 2, 3). The problem (1.1)-(1.5) is called the problem of pulse-free flow of the problem of flow around a self-propelled body.

It is well known that with prescribed functions $\mathbf{w}$ and $\mathbf{g}$ satisfying some regularity condition, problem (1.1)-(1.4) has at least one solution [1, 3]. [For example, for this it is sufficient that $\mathbf{w} \in C^{2+\alpha}(\Sigma)$, $|x|\mathbf{g} \in L_2(\Omega)$; Σ is the Lyapunov surface with an index $\alpha \in (0, 1)$.] With quite small (in suitable mathematical values) functions $\mathbf{w}$ and $\mathbf{g}$ the solution of problem (1.1)-(1.4) is unique [1, 2]. Whence it follows that the problem of flow around a self-propelled body (1.1)-(1.5) is solvable generally speaking only with fulfillment of additional conditions for functions $\mathbf{w}$ and $\mathbf{g}$. Formulation of these conditions [in other words, a study of the question of the resolvability of problem (1.1)-(1.5)] is a very difficult problem. Currently the question of existence of a solution for the problem of pulse-free flow has only been solved positively in a Stokes approximation [4] (in [4] there is also a review of previous results for studying the problem of flow around a self-propelled body). In [5] an approximate solution is constructed for one of the variants of axisymmetrical problem (1.1)-(1.5) with small Reynolds numbers in the case when Σ is a sphere.

The aim of this work is construction of an asymptotic solution for (1.1)-(1.5) with $r = |x| \to \infty$ assuming that it exists. Some previous results of studying this problem in the case of flow around a body with an immobile impenetrable boundary (which corresponds to $\mathbf{w} = -\mathbf{e}_1$) are given in [6]. An estimate of the decrease in curl velocity with $r \to \infty$ (without separating the main term of the asymptotics) for solution of problem (1.1)-(1.5) with $\mathbf{g} = 0$ was obtained in [7].

2. Integral Presentation of the Solution. First we formulate smoothness conditions in relation to starting data for problem (1.1)-(1.4) which subsequently will be assumed to be fulfilled everywhere. Let $\Sigma \in C^{2+\alpha}$, $w \in C^{2+\alpha}(\Sigma)$, $g \in C^{\alpha}(\Omega)$, where $0 < \alpha < 1$ (function g is assumed to have a compact carrier). With these assumptions the theorem is correct for existence of a classical solution of problem (1.1)-(1.4) in a class of vector-functions $\mathbf{u}$, having a finite Dirichlet integral [3]:

$$\int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{u} \, dx < \infty. \tag{2.1}$$

It was established in [2] that any solution of this problem satisfying inequality (2.1) accepts the estimate

$$|\mathbf{u}(x)| \leqslant C r^{-1/2-\varepsilon} \ \text{for} \ r \to \infty \tag{2.2}$$

with positive constants ε and C. It was shown in [1] that in order to solve problem (1.1)-(1.4) obeying condition (2.2) an integral presentation is valid

$$\begin{aligned} \mathbf{u}(x) = &-\int_{\Omega} [2\mathbf{u}(y)\cdot\mathbf{u}(y)\cdot\nabla E(x-y) + \mathbf{g}(y)\cdot E(x-y)]\, dy + \\ &+\int_{\Sigma} \{\mathbf{u}(y)\cdot PE(x-y) - E(x-y)\cdot P\mathbf{u}(y) + \\ &+ 2[E(x-y)\cdot\mathbf{u}(y)][\mathbf{u}(y)+\mathbf{e}_1]\}\cdot\mathbf{n}\, d\Sigma_y. \end{aligned} \tag{2.3}$$

Here E(x) is the fundamental tensor of the Oseen system corresponding to (1.1). Its elements E_{ij} (i, j = 1, 2, 3) are governed by the equations

$$E_{ij} = \delta_{ij}\Delta v - \frac{\partial^2 v}{\partial x_i \partial x_j}, \quad v = -\frac{1}{8\pi}\int_0^s \frac{1-e^{-t}}{t}\, dt, \quad s = r - x_1. \tag{2.4}$$

Symbol PE(x) signifies a tensor of the third rank with elements

$$(PE)_{ijk} = -P_k\delta_{ij} + \frac{\partial E_{ik}}{\partial x_j} + \frac{\partial E_{jk}}{\partial x_i}, \quad P_k = \frac{1}{4\pi}\frac{\partial}{\partial x_k}\left(\frac{1}{|\mathbf{x}|}\right), \quad i, j, k = 1, 2, 3. \tag{2.5}$$

For elements of tensor E and their first derivatives with large values of $r = |x|$ there are estimates [1, 2]

$$|E_{ij}| \leqslant C\frac{1}{r}\frac{1-e^{-s}}{s}; \tag{2.6}$$

$$|\nabla E_{ij}| \leqslant C\left(\frac{1}{r^{3/2}}\frac{1-e^{-s}-se^{-s}}{s^{3/2}} + \frac{1}{r^2}\frac{1-e^{-s}}{s}\right). \tag{2.7}$$

Here and below, C (with or without indices) are different positive constants. In addition, an estimate is required of the second derivatives of function E_{ij}. By quite cumbersome calculations these estimates are obtained from presentation (2.4) in the form

$$\left|\frac{\partial^2 E_{ij}}{\partial x_k \partial x_l}\right| \leqslant C\left(\frac{1}{r^2}\frac{1-e^{-s}-se^{-s}}{s^{3/2}(s+1)^{1/2}} + \frac{1}{r^{5/2}}\frac{1-e^{-s}}{s(s+1)^{1/2}}\right) \equiv C\psi(x), \tag{2.8}$$

if $r \to \infty$ (i, j, k, ℓ = 1, 2, 3).

By proceeding from presentation (2.3) it was shown in [1] that any solution of problem (1.1)-(1.4) satisfying equality (2.2) permits separation of the main term of the asymptotics with large values of r:

$$\mathbf{u}(x) = \mathbf{F}\cdot E(x) + \zeta(x), \tag{2.9}$$

where $\mathbf{F}$ is a constant vector determined by Eq. (1.5), and $\zeta(x)$ is a residual term for which an estimate is obtained

$$|\zeta| \leqslant C r^{-3/2+\varepsilon}(s+1)^{-1+\varepsilon} \tag{2.10}$$

($\varepsilon > 0$ is arbitrarily small). From (2.9) and (2.10) it emerges that there is a paraboloidal region of the trail in direction e_1, within which $u = O(r^{-1})$. Outside any circular cone with an axis directed along e_1, $u = O(r^{-2})$. (An estimate of the value of $|\zeta|$ outside the trail may be refined, but it is not required by us.)

Assuming colinearity of vectors $\mathbf{F}$ and e_1 in [8], the following terms of the asymptotics were obtained for field $\mathbf{u}$, having the order $r^{-3/2}$ (this assumption is fulfilled in the case of axisymmetrical flow). In [7, 8] a study of the behavior of the velocity curl at large distances from the body around which there was flow, also showed that outside the trail the curl decreases by an exponential rule. In both [7] and in [8] it is assumed that $g = 0$, and in [8] it is additionally assumed that $w = -e_1$, which relates to the classical flow problem. For this case in [9] "double" asymptotics were plotted for the velocity field when $r \to \infty$ and Reynolds number tends toward zero.

Equation (2.9) means that at considerable distances from a towed body perturbation of the velocity field will be (with an accuracy to small high orders) the same as for an Oseen stream "flowing around" a concentrated force $\mathbf{F}$. The consequence of this situation is the fact that information about the shape of a towed body is rapidly forgotten with distance from it. It is shown below that the asymptotics for the velocity field in the problem of pulse-free flow are governed by the markedly greater number of functionals characterizing both the shape of the body and the method of realizing the self-propulsion regime.

3. Main Result. Here and below it is assumed that the solution of problem (1.1)-(1.4) satisfies the additional condition of self-propulsion (1.5). In addition, it is assumed that condition (2.1) is fulfilled, which, in view of what has been said above makes it possible to present $\mathbf{u}(x)$ in the form of (2.3). We break down function into the sum of three terms: $\mathbf{u} = \mathbf{I} + \mathbf{J} + \mathbf{N}$, where

$$\mathbf{N}(x) = -2\int_{\Omega} \mathbf{u}(y)\cdot\mathbf{u}(y)\cdot\nabla E(x-y)\,dy,$$

$$\mathbf{J}(x) = -\int_{\Omega} \mathbf{g}(y)\cdot E(x-y)\,dy + \int_{\Sigma} \{-E(x-y)P\mathbf{u}(y) + 2[E(x-y)\cdot\mathbf{u}(y)][\mathbf{u}(y)+\mathbf{e}_1]\}\cdot\mathbf{n}\,d\Sigma_y$$

and I is the surface integral from $\mathbf{u}(y)\cdot PE(x-y)\cdot\mathbf{n}$. On the basis of (2.5) it is possible to write it as

$$\mathbf{I}(x) = \int_{\Sigma} \left\{[\mathbf{u}(y)\cdot\mathbf{n}]\,\nabla\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} + \mathbf{u}(y)\cdot 2DE(x-y)\cdot\mathbf{n}\right\} d\Sigma_y.$$

Here $DE(x)$ is a tensor of the third rank with elements

$$(DE)_{ijk} = \frac{1}{2}\left(\frac{\partial E_{ik}}{\partial x_j} + \frac{\partial E_{jk}}{\partial x_i}\right), \quad i,\ j,\ k = 1,\ 2,\ 3$$

(summing in the second term of the preexponential expression is carried out for indices i and j). We shall successively estimate integrals $\mathbf{I}$, $\mathbf{J}$, and $\mathbf{N}$.

In order to estimate $\mathbf{I}(x)$ we note that due to condition (1.4) the main term of the asymptotics of the integral of the first term with $r \to \infty$ equals zero. The second term for these asymptotics has the power r^{-3} with $r \to \infty$ and it should be considered together with the main term of the asymptotics of the integral containing $DE(x-y)$ since the last integral outside the region of the trail also has the power $O(r^{-3})$. Considering the limitedness of surface Σ and by using a Taylor equation, we obtain

$$\mathbf{I}(x) = R : DE(x) + \mathbf{q}\cdot\nabla\left(\nabla\frac{1}{|\mathbf{x}|}\right) + \chi_1(x), \tag{3.1}$$

where R is a constant tensor of the second rank; $\mathbf{q}$ is a constant vector; χ_1 is a residual term. Vector $\mathbf{q}$ and elements R_{ij} of tensor R in view of (1.2) are calculated explicitly:

$$\mathbf{q} = -\frac{1}{4\pi}\int_{\Sigma}(\mathbf{w}\cdot\mathbf{n})\,\mathbf{y}\,d\Sigma, \quad R_{ij} = 2\int_{\Sigma} w_i n_j\,d\Sigma, \quad i,\ j = 1,\ 2,\ 3.$$

The expression R:DE means convolution with respect to indices i and j. Residue χ permits the estimate

$$|\chi_1(x)| \leqslant C\psi(x) \text{ for } r \to \infty \tag{3.2}$$

[$\psi(x)$ is a function standing in the right-hand part of inequality (2.8)]. It is noted that in the zone of the trail it is determined by the inequality $s \equiv r - x_1 \leq C$, $\psi = O(r^{-2})$; outside any cone with axis $\mathbf{e}_1$, $\psi = O(r^{-4})$ when $r \to \infty$.

By considering the integral $\mathbf{J}(x)$ we can be sure that the main of its asymptotics with large r according to equality (1.5) vanishes to zero. In order to separate the following terms and estimate the residue again we use a Taylor equation and inequality (2.8), and we also consider the finiteness of function $\mathbf{g}$. As a result of this we arrive at the relationship

$$\mathbf{J}(x) = (Q_1 + Q_2) : \nabla E(x) + \chi_2(x). \tag{3.3}$$

Here Q_1 and Q_2 are constant tensors of the second rank; $\nabla E(x)$ is a tensor of the third rank with elements $(\nabla E)_{ijk} = \partial E_{jk}/\partial x_i$ (i, j, k = 1, 2, 3); χ_2 is a residual term permitting in the same way as χ_1, estimate (3.2). Summing in convolution $Q_\ell : \nabla E(x)$ (ℓ = 1, 2) is carried with respect to indices i and j. Elements of tensor Q_1 may be calculated a priori:

$$(Q_1)_{ij} = \int_\Omega y_i g_j dy - 2\int_\Sigma y_i w_j \sum_{l=1}^{3} (w_l + \delta_{1l})\, n_l d\Sigma.$$

In contrast to this the elements of tensor Q_2 are functionals from the solution of problem (1.1)-(1.5). They are calculated by the equation

$$(Q_2)_{ij} = \int_\Sigma y_i \sum_{l=1}^{3}\left(-p\delta_{jl} + \frac{\partial u_j}{\partial x_l} + \frac{\partial u_l}{\partial x_j}\right) n_l d\Sigma.$$

The estimate for integral $\mathbf{N}(x)$ is based on the results of work in [8]. For this it is noted that, in view of (1.5) and (2.9), $\mathbf{u} = \zeta$, which entails equality (2.10) for function $|u(x)|$. The presence of estimates (2.7) and (2.10) makes it possible to use Theorem 2 from [8] in order to calculate the main part of the integral with respect to region Ω from expressions $u_i(y)u_j(y)\partial E_{ik}(x - y)/\partial y_j$ (i, j, k = 1, 2, 3) and to estimate the residual term. As a result of this with $r \to \infty$

$$\mathbf{N}(x) = Q_3 : \nabla E + \omega(x), \tag{3.4}$$

where Q_3 is a constant tensor of the second rank, and function $|\omega|$ satisfies the equality

$$|\omega(x)| \leqslant Cr^{-2+\varepsilon}(s+1)^{-1/2}, \tag{3.5}$$

in which the number $\varepsilon > 0$ may be taken as small as is convenient. It is noted that standing in the right-hand side of (3.5) the function decreases more slowly than $\psi(x)$ with distance to infinity for any direction. Therefore, residual terms in Eqs. (3.1) and (3.3) will be subordinate compared with $\omega(x)$.

By combining equalities (3.1), (3.3), and (3.4), we arrive at the presentation sought for function u with large values of $|x| = r$:

$$\mathbf{u}(x) = R : DE(x) + Q : \nabla E(x) + \eta(x). \tag{3.6}$$

Here $Q = Q_1 + Q_2 + Q_3$, $\eta = \omega + \chi_1 + \chi_2$. From (3.2) and (3.5) and the remarks made above, it follows that

$$|\eta(x)| \leqslant C_0 r^{-2+\varepsilon}(s+1)^{-1/2} \text{ for } r \to \infty. \tag{3.7}$$

It is noted that in Eq. (3.6) there is no term $\mathbf{q}\cdot\nabla(\nabla|x|^{-1})$ as an extraordinal term; in the zone of the trail it is subordinate to the first two terms of the right-hand part of (3.6), and outside this zone it is subordinate to the last term.

We formulate the main result of the present work. Let u , p be the solution of problem (1.1)-(1.4) from class (2.1) satisfying additional condition (1.5). Then with $r \to \infty$ the asymptotic presentation $\mathbf{u}(x)$ in the form (3.6) is valid in which R and Q are constant tensors of the second rank. Elements of tensor R are expressed explicity in terms of the prescribed function $w(x)$. Function $\eta(x)$ permits estimate (3.7) in which ε and C_0 are positive constants; ε is arbitrarily small, and $s = r - x_1$.

In view of the presentation of (3.6), a series of unsolved questions arises. We have not yet set out explicit expressions in terms of function $\mathbf{u}$ for elements of tensor Q_3 figuring in (3.4) and entering into the sum comprising Q. It will be natural to assume that

$$(Q_3)_{ij} = -2\int_\Omega u_i(y)\,u_j(y)\,dy, \quad i,\ j=1,\ 2,\ 3.$$

(In the following section it is established that these integrals converge.) Furthermore, from (2.7) and (3.7) it emerges that outside any cone with axis $\mathbf{e}_1$ the first two terms in the right-hand part of (3.6) decrease more rapidly than the last. [There is a basis for assuming that in this region the index of $-2+\varepsilon$ of the power of r in inequality (3.7) may be substituted by $-5/2$, although proof of this assumption requires very detailed study of the integral N(x).] However, in the region of the trail $s \le C$ the function $\eta(x)$ which is most interesting from a physical viewpoint is the valid value of the residual term in Eq. (3.6).

By comparing asymptotic presentations (2.9) with $\mathbf{F}\neq 0$ and (3.6) we see that in the paraboloidal region of the trail $\mathbf{u} = O(r^{-1})$ in the first case and $\mathbf{u} = O(r^{-3/2})$ in the second. Outside any cone with $\mathbf{e}_1,\mathbf{u} = O(r^{-2})$ for a towed body ($\mathbf{F}\neq 0$) and $\mathbf{u} = O(r^{-5/2+\varepsilon})$ for a self-propelled body. Thus, there is a more rapid decrease in velocity perturbation at a considerable distance from a self-propelled body. In addition, the asymptotic behavior of velocity with pulse-free flow around a body is characterized by much greater variety than in the classical case. It is recalled that in the problem of flow around a towed body with an immobile impenetrable boundary in the absence of external mass forces the main term of the asymptotics is determined by the single vector F (and for an axisymmetrical flow regime by the single scalar vector F_1). As shown in [1], this vector is proportional to the resistance force acting on the body from the direction of the liquid.

Presentation (3.6) means that (at least in the region of the trail) the main terms of the asymptotics for the velocity field in the problem of pulse-free flow are characterized by eighteen parameters, i.e., elements of tensors R and Q. In the axisymmetrical case, the number of parameters decreases to eight. Identification of elements for tensor Q, which are some functionals from solving problem (1.1)-(1.5), is one of the most important questions in the group being considered.

4. Finiteness of the Energy Integral. We note one of the paradoxical results connected with the classical problem of flow for Navier–Stokes equations. In system (1.1) let $g = 0$ and in condition (1.2) $\mathbf{w} = -\mathbf{e}_1$ (the latter means immobility and impenetrability for the boundary of the body). Then for any solution $\mathbf{u}$, p of problem (1.1)-(1.4) satisfying condition (2.1), $\int_\Omega |\mathbf{u}|^2\,dx = \infty$ (confirmation of a similar type with respect to energy of perturbed movement in the problem of viscous flow was established for the first time in [10]).

It is intuitively clear that a self-propelled body cannot make such a considerable perturbation in the stream. The proper accurate formulation is as follows. Let $\mathbf{u}$, p be the solution of problem (1.1)-(1.4) satisfying additional conditions (1.5) and (2.1). Then

$$W = \frac{1}{2}\int_\Omega |\mathbf{u}|^2\,dx < \infty. \tag{4.1}$$

As already noted, condition (2.1) entails the correctness of presentation (2.9) for any solution of problem (1.1)-(1.4). Additional condition (1.5) means that function $\mathbf{u}(x)$ itself satisfies inequality (2.10). The required proof of (4.1) will be shown if it is established that the function standing in the right-hand part of (2.10) is quadratically summable in region $r \ge 1$ with quite small $\varepsilon > 0$. The latter for $\varepsilon < 1/2$ is almost obvious:

$$\int_1^\infty\int_0^\pi\int_0^{2\pi} r^{-3+2\varepsilon}\,[r(1-\cos\theta)+1]^{-2+2\varepsilon}\,r^2\sin\theta\,dr\,d\theta\,d\varphi =$$

$$= 2\pi \int_1^{\infty} r^{-2+2\varepsilon} \left[\int_0^{2r} (z+1)^{-2+2\varepsilon}\, dz \right] dr < \infty$$

(r, θ, and φ are spherical coordinates). Therefore, $W < \infty$.

The capacity for solution of the problem of pulse-free flow expressed by inequality (4.1) separate this solution from all possible solutions of problem (1.1)-(1.4) if functions **w** and **g** in its formulation are varied. This capacity may be used in order to study the question of existence of a solution for problem (1.1)-(1.5) if it is considered as an optimization problem.

Let number $N > 0$ be so great that surface Σ lies strictly within the sphere $\Sigma_N : |x| = N$. We designate in terms of Ω_N a region included between surfaces Σ and Σ_N in terms of $\mathbf{u}^N$ and p^N, i.e., solution of system (1.1) in region Ω_N satisfying condition (1.2) and

$$\mathbf{u}^N|_{\Sigma_N} = 0. \tag{4.2}$$

If there is fulfillment of relationship (1.4) and the regularity conditions in relation Σ, **w**, and **g**, listed at the beginning of Sec. 2, then problem (1.1), (1.2), (4.2) has at least one classical solution. It is clear that for this solution the energy integral $W_N = \frac{1}{2} \times \int_{\Omega_N} |\mathbf{u}^N|^2\, dx$ is finite.

Now we direct N toward infinity. Family $\{\mathbf{u}^N\}$ is compact in the distance function generated by the Dirichlet integral with fixed **w** and **g** [3]. However, generally speaking, $W_N \to \infty$ with $N \to \infty$. It is proposed to find a pair of functions $\mathbf{w}^N$ and $\mathbf{g}^N$, which yield a minimum functional W_N with a given finite N. If it is possible to provide limitedness for series W_N with $N \to \infty$ and to establish compactness in the appropriate distance value of family $\{\mathbf{w}^N, \mathbf{g}^N\}$, then limiting element **w**, **g** will determine the solution of problem (1.1)-(1.4) satisfying condition (4.1). Whence it is easy to conclude that this solution of **u**, p satisfied the self-propulsion condition (1.5).

5. Turbulent Flow Regime.

We study stationary turbulent flow around a self-propelled body. It appears that, in this case, it is possible to obtain some information about the behavior of velocity at a distance from the body on the basis of Sec. 3.

We shall consider Eq. (1.1) as a Reynolds equation for average velocities and pressure for stationary turbulent flow for which we retain the previous notations **u**(x) and p(x). The density of external forces figuring in Eqs. (1.1) has the form

$$\mathbf{g} = -2 \operatorname{div} \Pi,$$

where Π is the tensor for Reynolds stresses with elements $\Pi_{ij} = \overline{u_i' u_j'}$, u_i' is the pulsation component of the i-th component of the velocity vector, i, j = 1, 2, 3; a line above means the operation of averaging.

As is well known, at a considerable distance from a body turbulence degenerates. Therefore, it is natural to assume that functions Π_{ij} and their derivatives decrease rapidly with $r \to \infty$. In [11, Para. XIV] on the basis of the widespread hypothesis about self-modeling and additional ideas, expressions were obtained for longitudinal velocity and characteristic Reynolds stress in axisymmetrical turbulent flow behind a self-propelled body:

$$u_1 = x_1^{-4/5} g\left(\frac{x_2}{x_1^{1/5}}\right), \quad \Pi_{12} = x_1^{-8/5} h\left(\frac{x_2}{x_1^{1/5}}\right). \tag{5.1}$$

Here x_1 and x_2 are axial and radial coordinates of a cylindrical system; u_1 is deviation of the dimensionless average axial component of velocity from unity. In (5.1) it is assumed that $x_1 > 0$. Functions g and h decrease exponentially with $\eta = x_2/x_1^{1/5} \to \infty$. [Equation (5.1) has a limited field of applicability. On the one hand, $x_1 > 0$ should be sufficiently large that it would be possible to develop downwards through the flow a self-modeling movement regime, and, on the other hand, with very large values $x_1 > 0$ the turbulent nature of flow should vary in a laminary way. With large negative values of x_1 it should be generally assumed that $\Pi_{ij} = 0$.]

We assume that with $r \to \infty$ the inequality

$$|\Pi_{ij}| \leqslant Cr^{-2+\varepsilon}(s+1)^{-2+\varepsilon}, \quad |\nabla \Pi_{ij}| \leqslant Cr^{-5/2+\varepsilon}(s+1)^{-2+\varepsilon} \tag{5.2}$$

is fulfilled with $\varepsilon \in (0, 1)$ $(i, j = 1, 2, 3)$. Then, in order to solve the unclosed system (1.1) with conditions (1.2)-(1.4) when $\mathbf{g} = -2\operatorname{div}\Pi$ it is possible to obtain an integral presentation similar to (2.3) in which the second of the volumetric integrals is substituted by $\mathbf{N}_\Pi(x) = -2\int\limits_\Omega \Pi(y):\nabla E(x-y)\,dy$. Existence of inequality (5.2) makes it possible to find a presentation similar to (3.4) for function $\mathbf{N}_\Pi$. Further reasoning is similar to that provided in Sec. 3. It is possible to formulate it as follows.

Let elements Π_{ij} of the Reynolds stress tensor be satisfied by inequality (5.2) with $r \to \infty$. We assume that the self-propulsion condition $\int\limits_\Sigma [P\mathbf{u}\cdot\mathbf{n} - 2\mathbf{u}(\mathbf{u}+\mathbf{e}_1)\cdot\mathbf{n} - 2\Pi\cdot\mathbf{n}]d\Sigma = 0$ is fulfilled. Then for the average velocity vector $\mathbf{u}(x)$ asymptotic expression (3.6) is valid with $r \to \infty$ in which R and Q are constant tensors of the second rank.

It is emphasized that we arrive at this conclusion without drawing on any hypotheses of a semi-empirical nature. On the other hand, the values of Eq. (3.6) for turbulent flow at a distance from a self-propelled body should not be exaggerated. Evidently the region of its applicability commences from those distances at which the term $\operatorname{div}\Pi$ in order of value becomes less than $\partial\mathbf{u}/\partial x_1$.

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